CHEMNITZ UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Seminar Vector Valued Functions

Sequence Spaces

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Definition 1 (Norm)
Given a vector space E over $\mathbb{K}$. A norm *on E is a map $\|\cdot\| : E \mapsto \mathbb{R}_+$ that satisfies the following conditions:*

(N1) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K},\ x \in E$.

(N2) $\|x+y\| \leq \|x\| + \|y\|$ for all $x,y \in E$. (triangle inequality)

(N3) $\|x\| = 0$ if and only if $x = 0$.

The pair $(E, \|\cdot\|)$ is then called a normed space.

Definition 2 (Sequence space)
The space of all sequences ω is an ∞-dimensional space over $\mathbb{K}$.
A normed sequence space λ *is a subspace of ω including the sequence $e_n := (\delta_{j,n})_{j\in\mathbb{N}}\ \forall n \in \mathbb{N}$ with a norm $\|\cdot\|$.*

Obviously, the space of all *finite sequences*

$$\varphi := \left\{ x \in \omega : x = \sum_{j=1}^{N} \lambda_j e_j, \lambda_j \in \mathbb{K}, N \in \mathbb{N} \right\}$$

is a subspace of all normed sequence spaces.

Example 3
Space of all bounded sequences with supremum norm:

$$l_\infty := \{x \in \omega : \|x\|_\infty := \sup_{j\in\mathbb{N}} |x_j| < \infty\}$$

Space of all convergent sequences:

$$c := \{x = (x_n) \in \omega : x_n \to y, y \in \mathbb{K}\} \subset l_\infty$$

Space of all sequences convergent to 0:

$$c_0 := \{x = (x_n) \in \omega : x_n \to 0\} \subset c$$

Definition 4 (l_p)

$$l_p := \left\{ x \in \omega : \|x\|_p := \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} < \infty \right\}$$

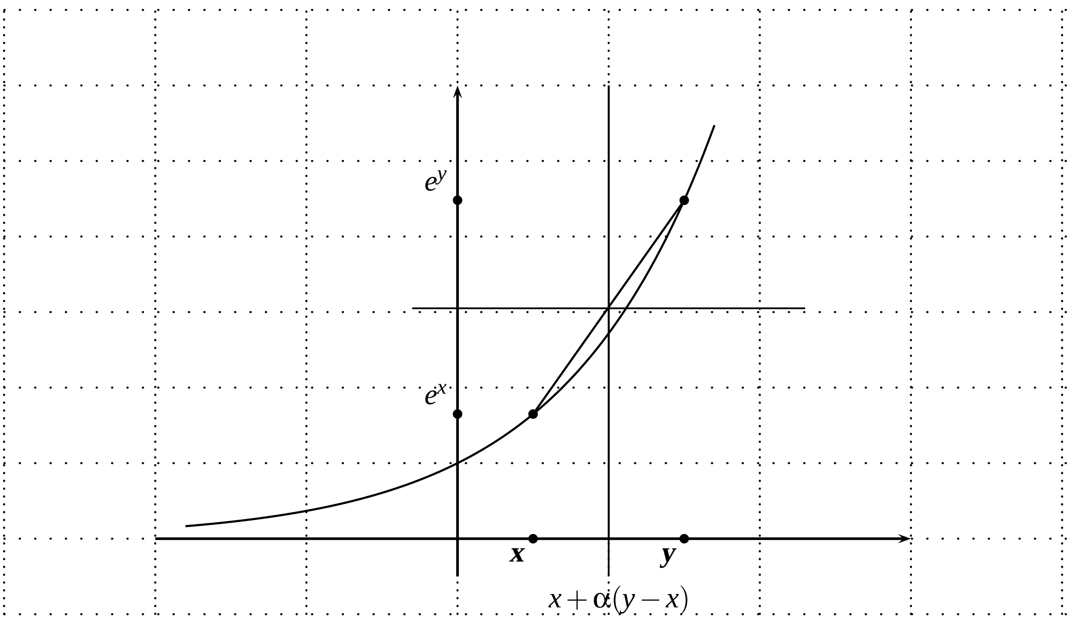
To show that l_p is a normed sequence space for $1 \leq p < \infty$, first:

Definition 5 (convex)
A map $f : \mathbb{R}^m \mapsto \mathbb{R}$ *is called* convex *if* $\forall x, y \in \mathbb{R}^m$, $\forall \alpha \in [0,1]$*:*

$$f(x+\alpha(y-x)) \leq f(x)+\alpha(f(y)-f(x))$$

Example 6
The exponential function e^x *is convex:*

$$e^{x+\alpha(y-x)} \leq e^x+\alpha(e^y-e^x)$$



Proposition 7 (Hölder inequality)
Let $p, q \in (1,\infty)$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. Then for* $x \in l_p$ *and* $y \in l_q$

$$(x_j y_j)_{j\in\mathbb{N}} \in l_1 \quad \text{and} \quad \sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \|y\|_q$$

Proof: *For $a,b>0$, set $A := p\ln a$, $B := q\ln b$. The exponential function e^x is convex, thus:*

$$\begin{aligned}\exp\left(\frac{A}{p}+\frac{B}{q}\right) &= \exp\left(B+\frac{1}{p}(A-B)\right)\\ &\le \exp(B)+\frac{1}{p}(\exp(A)-\exp(B))\\ &= \frac{1}{p}\exp(A)+\frac{1}{q}\exp(B)\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}a\cdot b &= \exp\left(\frac{1}{p}p\ln(a)\right)\exp\left(\frac{1}{q}q\ln(b)\right)\\ &\le \frac{1}{p}a^p+\frac{1}{q}b^q\end{aligned}$$

If $x\in l_p$ and $y\in l_q$ with $\|x\|_p = 1 = \|y\|_q$ then follows

$$\sum_{j=1}^{\infty}|x_j||y_j| \le \frac{1}{p}\sum_{j=1}^{\infty}|x_j|^p+\frac{1}{q}\sum_{j=1}^{\infty}|y_j|^q = \frac{1}{p}+\frac{1}{q} = 1$$

For $x\neq 0, y\neq 0$ given, use this for $x' := \frac{x}{\|x\|_p}$ and $y' := \frac{y}{\|y\|_q}$ and, thus, with multiplication with $\|x\|_p\|y\|_q$, it is

$$\sum_{j=1}^{\infty}|x_jy_j| \le \|x\|_p\|y\|_q$$

q.e.d.

Lemma 8 (Supremum formula)
Let p, $q\in(1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then for all $x\in\omega$:

$$\|x\|_p = sup\left\{|\sum_{j=1}^{\infty}x_jy_j| : y\in\varphi, \|y\|_q\le 1\right\}$$

Whereas, equality is meant in $[0,\infty]$.

Proof: *"$\ge$"*
Given $x\in\omega$ such that $\|x\|_p<\infty$, then follows of Hölder inequality:

$$\sum_{j=1}^{\infty}|x_jy_j| \le \|x\|_p \qquad \forall y\in\varphi \text{ with } \|y\|_q\le 1.$$

$$\Rightarrow sup \left\{ |\sum_{j=1}^{\infty} x_j y_j| : y \in \varphi, \|y\|_q \geq 1 \right\} \leq \|x\|_p$$

"$\leq$"

On the other hand, for $x \in \omega$, $x \neq 0$, let the supremum equal $C \in \mathbb{R}_+$. Choose $\lambda \in \omega$ with $|\lambda_j| = 1$ and $\lambda_j x_j = |x_j|$ $\forall j \in \mathbb{N}$. For sufficiently big $N \in \mathbb{N}$ $A := \left(\sum_{j=1}^{N} |x_j|^p\right)^{-\frac{1}{q}}$ exists. Define $y \in \varphi$ as $y_j = A\lambda_j |x_j|^{\frac{p}{q}}$ for $1 \leq j \leq N$ and $y_j = 0$ for $j > N$. Then, on the basis of the choice of A, the following applies:

$$\begin{aligned}
\|y\|_q &= \left(\sum_{j=1}^{N} |A\lambda_j |x_j|^{\frac{p}{q}}|^q\right)^{\frac{1}{q}} \\
&= A\left(\sum_{j=1}^{N} |\lambda_j|^q |x_j|^p\right)^{\frac{1}{q}} \\
&\overset{|\lambda|=1}{=} A\left(\sum_{j=1}^{N} |x_j|^p\right)^{\frac{1}{q}} = 1
\end{aligned}$$

Due to the fact that $p = 1 + \frac{p}{q}$, it follows that $\forall N \in \mathbb{N}$:

$$\begin{aligned}
C &\geq |\sum_{j=1}^{N} x_j y_j| = |\sum_{j=1}^{N} A \underbrace{x_j \lambda_j}_{=|x_j|} |x_j|^{\frac{p}{q}}| \\
&= A|\sum_{j=1}^{N} |x_j|^{1+\frac{p}{q}}| \\
&= A|\sum_{j=1}^{N} |x_j|^p| = \left(\sum_{j=1}^{N} |x_j|^p\right)^{1-\frac{1}{q}}
\end{aligned}$$

$\Rightarrow x \in l_p$ *and* $C \geq \|x\|_p \Rightarrow$ *equality.* *q.e.d.*

Proposition 9 (l_p normed space)
For $1 \leq p < \infty$, l_p is a normed sequence space.

Proof:
This is obvious for $l_1 := \{x \in \omega : \|x\|_1 := \left(\sum_{j=1}^{\infty} |x_j|\right) < \infty\}$

Consider $1 < p < \infty$:

Let $q := \frac{p}{p-1}$, *thus* $\frac{1}{p} + \frac{1}{q} = 1$. *Hence, for any* $x, y \in l_p$ *and* $\forall z \in \omega$ *(including all* $z \in \varphi$) *with* $\|z\|_q \leq 1$ *applies:*

$$|\sum_{j=1}^{N}(x_j + y_j)z_j| \overset{\substack{\text{triangle}\\\text{inequality}}}{\leq} |\sum_{j=1}^{N} x_j z_j| + |\sum_{j=1}^{N} y_j z_j| \qquad \forall N \in \mathbb{N}$$

For $N \to \infty$ *then follows:*

$$|\sum_{j=1}^{\infty}(x_j + y_j)z_j| \leq |\sum_{j=1}^{\infty} x_j z_j| + |\sum_{j=1}^{\infty} y_j z_j| \leq \|x\|_p + \|y\|_p.$$

The supremum formula yields

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \; \forall \, x, y \in l_p.$$

q.e.d.

Lemma 10 (Convergence of norm)
Let $q \in [1, \infty)$, $x \in l_q$, *then:*

$$\|x\|_p \to \|x\|_\infty := \sup_{j \in \mathbb{N}} |x_j| \text{ for } p \to \infty$$

Rationale: *It is able to show that*

$$\|x\|_\infty \leq \|x\|_p \leq \|x\|_\infty + \varepsilon \quad \forall p \geq p_0, p_0 = p_0(\varepsilon), \forall \varepsilon > 0$$

$$\Rightarrow \|x\|_p \to \|x\|_\infty \text{ for } p \to \infty.$$

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Corollary 11 (Extended supremum formula)
The supremum formula is also true for $p = \infty$ *and* $p = 1$*:*

$$\|x\|_\infty = sup \left\{ |\sum_{j=1}^{\infty} x_j y_j| : y \in \varphi, \|y\|_1 \leq 1 \right\}$$

$$\|x\|_1 = sup \left\{ |\sum_{j=1}^{\infty} x_j y_j| : y \in \varphi, \|y\|_\infty \leq 1 \right\}$$

Proof: *Consider $p \to \infty$ with $q = \frac{p}{p-1}$ and $q \to \infty$ with $p = \frac{q}{q-1}$, respectively.* *q.e.d.*

Definition 12 (Dual space of a sequence space)
For normed sequence spaces λ and μ, "$\mu = \lambda'$" is written if:
(D1) For each $y \in \mu$ and $x \in \lambda$ the series

$$\sum_{j=1}^{\infty} x_j y_j =: y(x)$$

is convergent and defines an element $y(\cdot)$ in λ', with $\|y(\cdot)\|_{\lambda'} = \|y\|_{\mu}$.
(D2) For each $\eta \in \lambda'$, a $y \in \mu$ exists with $y(\cdot) = \eta$.
μ is then called the dual space *of λ.*

So the notation $\mu = \lambda'$ means, that the map $y \mapsto y(\cdot)$ is an isometric isomorphsm between μ und λ'.

Reminder:

$$\|\eta\|_{\lambda'} := \sup\{|\eta(x)| : x \in \lambda, \|x\|_{\lambda} \leq 1\}$$

Proposition 13
Let $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $l'_p = l_q$. Furthermore $c'_0 = l_1$ and $l'_1 = l_\infty$.

Proof: *"$l'_p = l_q$":*
(D1) For $y \in l_q$ and $x \in l_p$ the absolut convergency of the series

$$y(x) = \sum_{j=1}^{\infty} x_j y_j$$

as well as the estimate

$$|y(x)| \leq \|x\|_p \|y\|_q, \text{ d. h. } \|y(\cdot)\|_{l'_p} \leq \|y\|_q.$$

results from the Hölder inequality. With the supremum formula, the following applies (after interchange of p and q):

$$\begin{aligned}\|y\|_q &= \sup\left\{ \left|\sum_{j=1}^{\infty} x_j y_j\right| : x \in \varphi, \|x\|_p \leq 1 \right\} \\ &\leq \sup\{|y(x)| : x \in l_p, \|x\|_p \leq 1\} \\ &= \|y(\cdot)\|_{l'_p}.\end{aligned}$$

$\Rightarrow \|y(\cdot)\|_{l'_p} = \|y\|_q$

(D2) For $\eta \in l'_p$ *define* $y \in \omega$ *as* $y_j = \eta(e_j)$, $j \in \mathbb{N}$. *Then for* $x \in l_p$ *this leads to*

$$x = \lim_{n\to\infty} \left(\sum_{j=1}^{n} x_j e_j \right) \quad in\ l_p$$

Therefore, $\eta \in l'_p$ *implicates*

$$\eta(x) = \lim_{n\to\infty} \eta \left(\sum_{j=1}^{n} x_j e_j \right) = \lim_{n\to\infty} \left(\sum_{j=1}^{n} x_j \eta(e_j) \right) = \sum_{j=1}^{\infty} x_j y_j = y(x).$$

$\Rightarrow y(\cdot) = \eta$

Hence, it follows, as above, that $\|y\|_q \le \|\eta\|_{l'_p} < \infty$, *consequentially* $y \in l_q$.

"$c'_0 = l_1$":

(D1) For $y \in l_1$ *and* $x \in c_0$, *the following applies:*

$$\begin{aligned} |y(x)| &\le \sum_{j=1}^{\infty} |x_j y_j| \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} |x_j y_j| \\ &\le \lim_{n\to\infty} \sum_{j=1}^{n} \left(\left(\sup_{k\in\mathbb{N}} |x_k| \right) |y_j| \right) \\ &= \left(\sup_{k\in\mathbb{N}} |x_k| \right) \lim_{n\to\infty} \sum_{j=1}^{n} |y_j| \\ &= \left(\sup_{k\in\mathbb{N}} |x_k| \right) \sum_{j=1}^{\infty} |y_j| \\ &= \|x\|_{c_0} \|y\|_{l_1} \end{aligned}$$

Thus, the series is absolutely convergent and defines an element $y(\cdot) \in c'_0$ *with* $\|y(\cdot)\|_{c'_0} \le \|y\|_{l_1}$.

$\|y(\cdot)\|_{c'_0} \ge \|y\|_{l_1}$ *follows from the extended supremum formula as above.*

(D2) Along the lines of "$l'_p = l_q$ *(D2)*"

"$l'_1 = l_\infty$": *Along the lines of* "$c'_0 = l_1$" *q.e.d.*

Definition 14 (Banach space)

A complete normed vector space E is called a Banach space.

(Complete: Every Cauchy sequence in E is convergent within E.)

To show that l_p (for $p \in [1,\infty]$), c and c_0 are Banachspaces, some considerations are necessary beforehand.

Lemma 15
Let E, F be normed spaces. If F is a Banach space then

$$L(E,F) := \{A : E \mapsto F | \text{ A linear, continuous map}\}$$

is a Banach space as well.

Lemma 16 (Completeness of c, c_0)
c_0 and c are complete.

Proof: *Completeness of c:*
Let $((x_n)^m)$ be a Cauchy sequence in c
$\Rightarrow\ \exists\ (y_n) \in l_\infty$ *with* $(x_n)^m \to (y_n)$ *in* l_∞*, for* l_∞ *is dual space and, thus, complete according to lemma 15* $(c \subset l_\infty)$.

To check $(y_n) \in c$:
It is essential that $\forall \varepsilon \geq 0\ \exists m_0 \in \mathbb{N}$ *so that* $\forall m \geq m_0$:

$$\|x_n^m - y_n\|_\infty = \sup_n |x_n^m - y_n| \leq \frac{\varepsilon}{2}$$

Furthermore, $\forall m\ \exists\ \tilde{x}^m : \forall \varepsilon\ \exists n_0 \in \mathbb{N}$ *so that* $\forall n \geq n_0$

$$|x_n^m - \tilde{x}^m| \leq \frac{\varepsilon}{2} \qquad \text{because } (x_n)^m \text{ is in c.}$$

$$\begin{aligned} |y_n| &= |y_n - x_n^m + x_n^m - \tilde{x}^m + \tilde{x}^m| \\ &\leq |y_n - x_n^m| + |x_n^m - \tilde{x}^m| + |\tilde{x}^m| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + |\tilde{x}^m| \qquad \text{for } n \geq n_0,\ m \geq m_0 \end{aligned}$$

Vice versa:

$$\begin{aligned} |\tilde{x}^m| &= |\tilde{x}^m - x_n^m + x_n^m - y_n + y_n| \\ &\leq |\tilde{x}^m - x_n^m| + |x_n^m - y_n| + |y_n| \\ &\leq \varepsilon + |y_n| \end{aligned}$$

$\Rightarrow\ (y_n)$ *convergent* $\Rightarrow (y_n) \in c$.
Completeness of c_0:
Let $((x_n)^m)$ *be a Cauchy sequence in* c_0
$\Rightarrow\ \exists (y_n) \in l_\infty$ *with* $(x_n)^m \to (y_n)$ *in* l_∞*, for* $c_0 \subset l_\infty$.
The evidence that (y_n) *converges to* 0 *is analogous.* *q.e.d.*

Definition 17 (Reflexivity)
Let $(E, \|\cdot\|_E)$ *be a Banach space. If the so-called* canonical inclusion

$$J : E \mapsto E'' \quad \text{with} \quad x \mapsto (x' \mapsto x'(x))$$

(i.e. $J(x)[x'] = x'(x)$*) is surjective, then* E *is called a* reflexive *Banach space.*

Corollary 18 (Sequence spaces are Banach spaces)
a) For $1 < p < \infty$*,* l_p *is a reflexive Banach space.*
b) $c_o,\ c,\ l_1$ *and* l_∞ *are Banach spaces.*
c) c_o *is not reflexive.*

Proof: *Completeness needs to be checked:*
For $p \in [1, \infty]$ *the spaces* l_p *are dual spaces according to proposition 13 and, thus, complete after lemma 15 (*$E = \lambda$*,* $F = \mathbb{R}$*,* $L(\lambda, \mathbb{R}$ *or* $\mathbb{C}) = \lambda'$*).*
c *and* c_0 *are complete according to lemma 16. Therefore, they all are complete normed vector spaces, thus Banach spaces.*

Reflexivity for $p \in (1, \infty)$*:*
Let $p, q \in (1, \infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*, then* $l'_p = l_q$ *and* $l'_q = l_p$ *as well as* $l''_p = l_p$ *apply. For* $x \in l_p$ *and* $y \in l_q$ *follows:*

$$J(x)[y] = y(x) \stackrel{!}{=} x(y)$$

$$y(x) = \sum_{j=1}^{\infty} x_j y_j = \sum_{j=1}^{\infty} y_j x_j = x(y)$$

$\Rightarrow J$ *is surjective.* $\Rightarrow l_p$ *is reflexive.*

To show that c_0 *is not reflexive:*
For $x \in c_0$ *and* $y \in l_1$ *results, in a similar manner, that* $J(c_0) = c_0 \neq l_\infty$*. So here* J *is not surjective.* $\Rightarrow c_0$ *is not reflexive.* *q.e.d.*

Remark 19
The Banach spaces $c,\ l_1$ *and* l_∞ *are not reflexive.*

Rationale:
The closed subspace of a reflexive Banach space is reflexive as well.
c_0 *is a closed subspace of* c *and* l_∞ *and is not reflexive. Thus,* c *and* l_∞ *cannot be reflexive.*
A Banach space is reflexive if and only if its dual space is reflexive.
Since c_0 *is not reflexive, its dual space* $l_1 = c'_0$ *is not reflexive as well.* //

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